JOURNAL OF ALGEBRA 107, 17–27 (1987)

E-Transitive Torsion-Free Abelian Groups

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Communicated by I. N. Herstein

Received February 1, 1985

1. INTRODUCTION

A torsion-free abelian group G is said to be strongly homogeneous if, given any two pure rank-one subgroups X and Y of G, there exists an automorphism α of G such that $X\alpha = Y$. The torsion-free strongly homogeneous groups of finite rank were completely characterized by Arnold [1]. Reid [12] considers strongly homogeneous groups of arbitrary rank, and the most detailed results on such groups known so far were obtained by Krylov [9].

In this note we consider an apparent generalization: the torsion-free abelian group G is said to be E-transitive if, given any two pure rank-one subgroups X and Y of G, there exists an endomorphism ε of G such that $X\varepsilon = Y$. Clearly, every strongly homogeneous group is E-transitive. We were unable to prove the converse. Since the E-transitive groups relate to the strongly homogeneous ones as Irving Kaplansky's fully transitive R-modules relate to the transitive R-modules [8, p. 58] (replace "Ulm sequence" by "characteristic" and observe Remark 1 of [9, p. 216]) we take solace in Kaplansky's statement that "it is by no means clear that full transitivity implies transitivity" [8, p. 58].

The purpose of this note is to show that Krylov's results on strongly homogeneous groups [9] hold true for E-transitive groups as well, and to expand and augment some of these results.

Let C denote the center of the endomorphism ring $E = E(G)$ of G. We show that G is E-transitive if and only if C is a strongly homogeneous integral domain (in fact, a P.I.D.), G is a torsion-free C-module, and E acts transitively on the pure rank-one C-submodules of G (3.4). As is the case for the class of strongly homogeneous groups, a reduction theorem allows

* This research was supported in part by a University of Houston–University Park Research Enabling Grant.

0021-8693/87 $3.00

the restriction to the E-cyclic ones (3.1). All C-submodules of at most countable C-rank of an E-cyclic E-transitive group F are free (3.2) and, if F has a C-submodule of rank one whose additive endomorphisms lift to F, then C is an E-ring (4.3).

Two torsion-free E-transitive groups are isomorphic if and only if their types are equal and there exists a topological isomorphism between their endomorphism rings (3.8). Every topological automorphism of the endomorphism ring of an E-transitive group is inner (3.9). An E-transitive group with nonzero endomorphic image of countable C-rank is strongly homogeneous (4.2). Hence, for groups of at most countable C-rank, the notions of E-transitivity and strong homogeneity coincide.

2. Preliminaries

Throughout, G is a torsion-free abelian group with endomorphism ring $E = E(G)$. Mappings are written to the right. Notation and terminology for the most part are those of Fuchs [4, 5]. Following Reid [10, 11], we call G (strongly) irreducible if, for every nonzero fully invariant subgroup F of G, the quotient group G/F is a (bounded) torsion group. We have

Lemma 2.1. *If G is E-transitive then G is homogeneous and irreducible.*

Proof. For any two pure rank-one subgroups X and Y of G, $\mathrm{Hom}(X, Y) \neq 0$. By [2, p. 7, 1.3], X and Y are isomorphic. Let $0 \neq F$ be a fully invariant subgroup of G and let $x \in G$. If $0 \neq y \in F$, there exists $\varepsilon \in E$ such that $\langle y \rangle_* \varepsilon = \langle x \rangle_*$. Hence $x \in \langle F\varepsilon \rangle_* \subseteq \langle F \rangle_*$. Since x was arbitrary, $\langle F \rangle_* = G$.

We consider G as embedded into its divisible hull V. For H a subgroup of V and $A \leqslant Q$ a subgroup of the rationals, we let

$$AH = \{ah \in V \mid a \in A, h \in H\}.$$

Then AH is a subgroup of V, and $AH \simeq A \otimes H$, the mapping $a \otimes h \mapsto ah$ being an isomorphism. We find it more straighforward to work with AH instead of the tensor product one reason being that $E(H)$ *is* a ring of linear transformations of V. We make use of this in

Proposition 2.2. *Let G be E-transitive and $0 \neq g \in G$. Let A be a subgroup of Q containing 1 such that $\chi_A(1) = \chi_G(g)$, and let $F = gE$. Then $G = AF$, and F is an E-transitive E-cyclic fully invariant subgroup of G of idempotent type. Moreover, $E(G) = E(F)$ as rings of endomorphisms of $V = QG$.*

Proof. Note that $\langle g \rangle_* = Ag$. Hence $AF \subseteq G$. If $0 \neq x \in G$ then $\langle x \rangle_* = \langle g \rangle_* \varepsilon = Ag\varepsilon$ for some $\varepsilon \in E$, proving $G \subseteq AF$, and $G = AF$. Clearly, $F = gE$ is fully invariant and every endomorphism of F extends to $AF = G$. Hence $E(G)|F = E(F)$. In order to verify that F is E-transitive, let X and Y be pure rank-one subgroups of F. Then AX and AY are pure rank-one subgroups of $AF = G$, and $AX \cap F = X$, $AY \cap F = Y$. Since G is E-transitive there exist ε and ϕ in E such that $(AX)\varepsilon = AY$ and $(AY)\phi = AX$. Thus

$$X\varepsilon = (AX \cap F)\,\varepsilon \subseteq AY \cap F = Y.$$

From

$$(AY)\,\phi\varepsilon = (AX)\,\varepsilon = AY$$

we deduce the existence of a rational number r such that $\phi\varepsilon \mid AY = r \cdot 1$. Since G is homogeneous, $G = rG$, and $F = rF$. Hence

$$Y\phi\varepsilon = (AY \cap F)\,\phi\varepsilon = r(AY \cap F) = AY \cap F = Y.$$

It follows that $X\varepsilon = Y$. Having verified that F is E-transitive, its homogeneity follows from 2.1. Finally, if the prime p divides g in F, $g = pg\varepsilon$ for some $\varepsilon \in E$ proving $g = p^n g\varepsilon^n \in p^n F$ for all $n \in N$. This completes the proof.

The concept of irreducibility for abelian groups introduced by Reid [10] allows one to invoke the power of Schur's Lemma and the Jacobson Density Theorem [7, Chap. II]: if G is irreducible, its divisible hull $V = QG$ is a simple QE-module and $D = \mathrm{Hom}_{QE}(V, V)$ is a division ring; moreover, QE is a dense subring of the ring $L = \mathrm{Hom}_D(V, V)$. Taking as neighborhood basis of zero in $E = E(G)$ the annihilators of finite subsets of G makes E into a topological ring which is Hausdorff [4, p. 221]; L is the completion of QE [7, p. 31].

For strongly homogeneous groups the following result is due to Krylov [9, p. 218, 4]. A ring with identity is said to be strongly homogeneous if every element is an integral multiple of a unit [9].

PROPOSITION 2.3. *Let G be E-transitive and let $C = \mathrm{Cent}\, E$ be the center of $E = E(G)$. Let $V = QG$ and $D = \mathrm{Hom}_{QE}(V, V)$. Then $D = QC$, and C is a strongly homogeneous principal ideal domain.*

Proof. Obviously, $QC \subseteq D$. To prove equality, let $\delta \in D$ and $0 \neq g \in G$. Since V/gE is torsion, there exists a natural number n such that $ng\delta \in gE$. Hence, for all $\varepsilon \in E$, $ng\varepsilon\delta = ng\delta\varepsilon \in gE\varepsilon \subseteq gE$. Thus $gE(n\delta) \subseteq gE$. By 2.2, $E(gE) = E$. It follows that $n\delta \in E$ and, consequently, $n\delta \in C$. This implies $\delta \in QC$. We have shown that $D = QC$. In particular, C is an integral domain, and every $\zeta \in C$ is a monomorphism. To show that C is strongly homogeneous let $0 \neq \zeta \in C$, let B be a pure rank-one subgroup of G, and let $W = \langle B\zeta \rangle_*$. Since $W \simeq B \simeq B\zeta \leqslant W$, there exists an integer n such that $B\zeta = nW$. Let X be any pure rank-one subgroup of G. Then $B\phi = X$ and $W\psi = X$ for some ϕ, $\psi \in E$. Hence

$$X\zeta = B\phi\zeta = B\zeta\phi = nW\phi \subseteq nG$$

and

$$nX = nW\psi = B\zeta\psi = B\psi\zeta \subseteq G\zeta.$$

Since X was arbitrary, it follows that $G\zeta = nG$. Let $\delta = n^{-1}\zeta \in QC = D$. Then $G\delta = G\zeta n^{-1} = G$ so that $\delta \in E \cap QC = C$, and δ is an automorphism. Thus $\zeta = n\delta$ is an integral multiple of a unit in C. By Remark 2 of [9, p. 217], C is a P.I.D.

For future reference we record

LEMMA 2.4. *Let C be a torsion-free strongly homogeneous principal ideal domain, let M and U be torsion-free right C-modules, and let U have C-rank one. Then*

(i) *For every Z-pure subgroup H of M, HC is a C-pure submodule of M.*

(ii) *For every submodule $S \neq 0$ of U, U/S is a torsion group.*

(iii) *If X and Y are Z-pure subgroups of U then there exists a unit $\gamma \in C$ such that $X\gamma = Y$.*

Proof. (i) Let $x \in M$, $h \in H$, $\zeta_i \in C$ such that $x\zeta_1 = h\zeta_2$ and $\zeta_1 \neq 0$. Since $\zeta_i = n_i\alpha_i$ for some integers n_i and units α_i in C, $n_1 x\alpha_1\alpha_2^{-1} = n_2 h \in H$. The Z-purity of H implies $x\alpha_1\alpha_2^{-1} \in H$ so that $x \in H\alpha_2\alpha_1^{-1} \subseteq HC$.

(ii) The quotient field of C is QC, and there exists a C-monomorphism ϕ: $U \to QC$ such that $C \subseteq S\phi$ [14, p. 35, 2.8].

(iii) Since U is C-isomorphic to a C-submodule of QC containing 1, it suffices to consider this case. Then $C \subseteq U$ and U/C is a torsion group. Let $0 \neq x \in X$, $0 \neq y \in Y$. Then there are m, $n \in N$ such that $mx \in C$ and $ny \in C$. Hence $mx = m'\alpha$, $ny = n'\beta$ for some m', $n' \in N$ and units α, $\beta \in C$. The purity of X and Y implies $\alpha \in X$, $\beta \in Y$. Let $\gamma = \alpha^{-1}\beta$. Then $\gamma \cdot 1$ is an automorphism of U. Hence, $X\gamma$ is pure in U, and

$$\beta = \alpha(\alpha^{-1}\beta) = \alpha\gamma \in X\gamma$$

implies

$$X\gamma = Y.$$

The proof of [9, p. 220, Theorem 2] can be adapted to show

PROPOSITION 2.5. *Let G and G' be two strongly irreducible torsion-free abelian groups, and let $\psi: E(G) \to E(G')$ be a topological ring isomorphism. Then $G \doteq G'$. In particular, if $0 \neq g \in G$, there exists $g' \in G'$ such that the map $g\varepsilon \to g'(\varepsilon\psi)$, $\varepsilon \in E$, is an isomorphism from $gE(G)$ onto $g'E(G')$.*

Proof. Put $E = E(G)$, $V = QG$, $D = \mathrm{Hom}_{QE}(V, V)$ and $L = \mathrm{Hom}_D(V, V)$, and let $E' V'$, D' and L' be their counterparts for G'. The topological isomorphism $\psi: E \to E'$ extends to an isomorphism from L to L' which we continue to denote by ψ. Let $0 \neq g \in G$. Then $V = gD \oplus V_1$. Let $\pi \in L$ such that $g\pi = g$ and $V_1\pi = 0$. Then $0 \neq \pi = \pi^2$. Put $\pi\psi = \pi'$. Then $V' = V'\pi' \oplus V_2$ with $V_2\pi' = 0$. Pick $0 \neq g' \in V'\pi' \cap G'$. Since π is primitive, so is π' which implies $V'\pi' = g'D'$. Define

$$\phi: \quad gE \to g'E'$$

by $(g\varepsilon)\phi = g'(\varepsilon\psi)$ for all $\varepsilon \in E$. Since G/gE and $G'/g'E'$ are groups of bounded order, the proof is completed once we show ϕ is an isomorphism. To verify ϕ is a map, assume $g\varepsilon = g\eta$ for some ε, $\eta \in E$. Then $0 = g(\varepsilon - \eta) = g\pi(\varepsilon - \eta)$ so that $\pi(\varepsilon - \eta) = 0$. Hence $\pi'(\varepsilon\psi - \eta\psi) = 0$ and

$$g'(\varepsilon\psi) = g'\pi'(\varepsilon\psi) = g'\pi'(\eta\psi) = g'(\eta\psi).$$

Obviously, ϕ is additive and maps onto $g'E'$. To show ϕ is monic, assume $g\varepsilon\phi = 0$, $\varepsilon \in E$. Then

$$0 = g'(\varepsilon\psi) = g'\pi'(\varepsilon\psi)$$

so that $0 = \pi'\varepsilon\psi = (\pi\varepsilon)\psi$. Hence $\pi\varepsilon = 0$ which implies $0 = g\pi\varepsilon = g\varepsilon$. The proof is completed.

3. THE MAIN RESULTS

The following theorem allows the reduction to E-cyclic groups and shows how to obtain E-transitive extensions.

THEOREM 3.1. *The following properties of the torsion-free abelian group G are equivalent.*

(i) *G is E-transitive.*

(ii) *$G = AF$ where F is an E-transitive group, and $A \leqslant Q$ is a subgroup of the rationals.*

(iii) $G = AF$ *where* $A \leqslant Q$ *and* F *is an E-transitive E-cyclic fully invariant subgroup of* G *of idempotent type such that, as rings of endomorphism of* QG, $E(F) = E(G)$.

Proof. By 2.2, (i) implies (iii) which in turn implies (ii). Assume (ii) and let X_i be pure rank-one subgroups of G, $i = 1, 2$. Then $X_i \cap F$ are pure rank-one subgroups of F, and $X_i = A(X_i \cap F)$, $i = 1, 2$ [2, p. 19, 1.2(a)]. Since F is E-transitive, $(X_1 \cap F)\phi = X_2 \cap F$, and ϕ induces an endomorphism in $G = AF$ mapping $X_1 = A(X_1 \cap F)$ onto $A(X_2 \cap F) = X_2$. The proof is completed.

A module M over a principal ideal domain R is said to be $\aleph_1$-free if M is a torsion-free R-module and every submodule of at most countable R-rank is free [4, p. 94]. Since Pontryagin's Theorem [4, p. 93, 19.1) holds for modules over P.I.D.s (and submodules of free R-modules are free), a torsion-free R-module is $\aleph_1$-free if all pure submodules of finite rank are free.

Krylov has shown that every E-cyclic strongly homogeneous group F is an $\aleph_1$-free module over $C = \text{Cent } E(F)$ [9, p. 217, Theorem 1]. His proof essentially works for E-transitive groups. We include it for the sake of completeness.

THEOREM 3.2. *Let* F *be an E-transitive E-cyclic torsion-free group and let* $C = \text{Cent } E(F)$. *Then* F *is an* $\aleph_1$*-free C-module.*

Proof. By 2.3, C is a P.I.D. and, by [6, Lemma 5], F is a torsion-free C-module. Let M be a pure submodule of F of finite rank k. To show M is free we induct on k. Assume $k = 1$ and let X be a pure rank-one subgroup of F contained in M. By 2.2, $F = gE(F)$ has idempotent type which implies

$$\langle g \rangle_* = g(C \cap Q).$$

Also, $X\varepsilon = \langle g \rangle_*$ for some $\varepsilon \in E(F)$. By 2.4(i), $M = XC$ so that

$$\begin{aligned} M\varepsilon = (XC)\,\varepsilon = X\varepsilon C &= \langle g \rangle_* C \\ &= g(C \cap Q)\, C = gC \end{aligned}$$

is cyclic. 2.4(ii) implies $M \simeq M\varepsilon = gC$ proving M is free. Suppose that $k \geqslant 1$ and C-submodules of rank $k-1$ are free. We invoke 2.3: let $V = QF$ and $E = E(F)$. Then $D = \text{Hom}_{QE}(V, V) = QC$, and QM is a D-submodule of V. If $0 \neq a \in M$, then $QM = aD \oplus V_1$, and $V = QM \oplus V_2$. Also, QE is a dense ring of D-linear transformations of V [7, p. 31]. Hence, there exists $\sigma \in QE$ such that $a\sigma = a$ and $V_1\sigma = 0$. Let $n \in N$ such that $n\sigma = \varepsilon \in E$. Then

$$M\varepsilon \subseteq (aD \oplus V_1)(n\sigma) = aDn\sigma = naD$$

which implies $M\varepsilon \subseteq F \cap aD$ is a C-submodule of F of rank one. Thus $M\varepsilon$ is free and $M \simeq \ker \varepsilon \oplus M\varepsilon$. By induction hypothesis, the C-module $\ker \varepsilon$ is free, and so is M.

Combining 3.2 with 2.2 and 2.3 we state the analogue to Krylov's Theorem 1 [9, p. 217].

COROLLARY 3.3. *Let G be an E-transitive torsion-free group and let $C = \operatorname{Cent} E(G)$. Then C is a strongly homogeneous principal ideal domain, G is a torsion-free C-module, and $G = AF$ where F is an $\aleph_1$-free C-submodule of G and A is a torsion-free rank-one group whose type equals to the type of G.*

We have the following characterization of E-transitive groups as modules over strongly homogeneous P.I.D.s.

THEOREM 3.4. *For the torsion-free abelian group G, the following properties are equivalent.*

(i) *G is E-transitive*

(ii) *$C = \operatorname{Cent} E(G)$ is a strongly homogeneous principal ideal domain, G is a torsion-free C-module and $E(G)$ acts transitively on the pure rank-one C-submodules of G.*

(iii) *G is a torsion-free module over a strongly homogeneous principal ideal domain R, and the ring $E_R(G)$ of R-endomorphisms of G acts transitively on the pure rank-one R-submodules of G.*

Proof. To derive (ii) from (i), observe 3.3, and let U_i be C-pure submodules of G of rank one. If X_i are rank-one pure subgroups of G contained in U_i, $i = 1, 2$, then $X_1\varepsilon = X_2$ for some $\varepsilon \in E$ and $U_i = X_iC$, $i = 1, 2$, by 2.4(i). Hence

$$U_1\varepsilon = X_1C\varepsilon = X_1\,\varepsilon C = X_2C = U_2,$$

proving (ii). Clearly, (ii) implies (iii). Assume (iii) and let X_i, $i = 1, 2$, be pure rank-one subgroups of G. Since G is torsion-free, R is a torsion-free ring and, by 2.4(i), X_iR are rank-one pure R-submodules of G. Hence $X_1R\theta = X_2R$ for some $\theta \in E_R(G)$, and, by 2.4 (ii), $\theta \mid X_1R$ is monic. It follows that $X_1\theta$ and X_2 are pure rank-one subgroups of U_2 so that $X_2 = X_1\theta\gamma$ for some unit $\gamma \in R$, by 2.4(iii). Since $\theta\gamma \in E(G)$ the proof is completed.

Remark 3.5. In 3.4, reeplacing $E(G)$ and $E_R(G)$ by the automorphism groups $A(G)$ and $A_R(G)$, yields the analogous characterization for strongly homogeneous groups (in the proof, pick ε and θ to be automorphisms).

For torsion-free modules over P.I.D.s, define separability as in [5, p. 117]. As a consequence of 3.5 we have

COROLLARY 3.6. *Every torsion-free separable homogeneous module over a torsion-free strongly homogeneous principal ideal domain is strongly homogeneous.*

We turn to endomorphism rings. In analogy to Krylov [9, p. 220, Theorem 2] we have

THEOREM 3.7. *Let G and G' be E-transitive torsion-free abelian groups and let A and A' be torsion-free groups of rank one whose types equal the types of G and G', respectively. Let $\psi: E(G) \to E(G')$ by a topological ring isomorphism. Then $A'G \simeq AG'$. Moreover, there exists an isomorphism $\Phi: A'G \to AG'$ such that, for all $\eta \in E(G)$, $\eta\psi = \Phi^{-1}\eta\Phi$.*

Proof. As in 2.2, let $G = BF$, $F = gE$, $E = E(G) = E(F)$, $B \leqslant Q$, and likewise $G' = B'F'$, $F' = g'E'$, $E' = E(G') = E(F')$, $B' \leqslant Q$. Then $A \simeq B$, $A' \simeq B'$ which implies $A'B \simeq AB'$. Let $\sigma: A'B \to AB'$ be an isomorphism. Note that E-cyclic irreducible groups are strongly irreducible [11, p. 43, Theorem 2]. Hence, by 2.5, there is an isomorphism

$$\phi: gE \to hE',$$

for some $h \in F'$, which is given by $g\varepsilon \mapsto h(\varepsilon\psi)$, $\varepsilon \in E$.

Since F' has idempotent type, $h = nh'$ for some $h' \in F'$ such that $\chi(h') = \chi(g')$. F' being E-transitive implies $g' = h'\theta$ for some $\theta \in E'$ so that $F' = g'E' = h'E'$. Hence, the map

$$\phi': \quad gE \to h'E',$$

defined by $g\varepsilon \mapsto h'(\varepsilon\psi)$, $\varepsilon \in E$, is an isomorphism from F onto F'. Define

$$\Phi: \quad A'G \to AG'$$

by $(a'bg\varepsilon)\,\Phi = (a'b)\,\sigma h'(\varepsilon\psi)$ for all $a' \in A'$, $b \in B$, $\varepsilon \in E$. Then Φ is an isomorphism and, for all $\eta \in E$,

$$\begin{aligned}(a'bg\varepsilon)\,\Phi(\eta\psi) &= [(a'b)\,\sigma h'(\varepsilon\psi)](\eta\psi)\\ &= (a'b)\,\sigma h'[(\varepsilon\eta)\,\psi]\\ &= (a'bg\varepsilon\eta)\,\Phi = (a'bg\varepsilon)(\eta\Phi).\end{aligned}$$

Hence $\Phi(\eta\psi) = \eta\Phi$ which implies $\eta\psi = \Phi^{-1}\eta\Phi$ as stated.

COROLLARY 3.8. *Two torsion-free E-transitive groups are isomorphic if and only if they have equal types and their endomorphism rings are topologically isomorphic. If this is the case, any topological isomorphism of the rings is induced by an isomorphism of the groups.*

Proof. Let G and G' be E-transitive of equal types and ψ: $E(G) \to E(G')$ a topological ring isomorphism. Pick $0 \neq g \in G$ and $g' \in G'$ of equal characteristic. By 2.2, $G = AF$, $G' = AF'$, and ψ: $E(F) \to E(G')$ is a topological ring isomorphism. Let $1 \in B \leqslant Q$ such that $\chi_B(1) = \chi_F(g)$. Then $BA = A$ and the type of B equals the type of F. Hence $AF = G$ and $BG' = BAF' = AF' = G'$. Apply 3.7.

COROLLARY 3.9. *Any topological automorphism of the endomorphism ring of a torsion-free E-transitive group is inner.*

4. CONSEQUENCES AND PROBLEMS

The central question we were unable to decide is

PROBLEM 1. Is every torsion-free E-transitive group strongly homogeneous?

The only E-transitive groups known to us are torsion-free homogeneous separable modules over strongly homogeneous P.I.D.s. By 3.6, these are strongly homogeneous. We pose

PROBLEM 2. Is every strongly homogeneous (E-transitive) torsion-free group a torsion-free homogeneous separable module over a strongly homogeneous principal ideal domain?

One criterion for separability is contained in

THEOREM 4.1. *Let G be an E-transitive group and let $C = \mathrm{Cent}\, E(G)$. Then G is a separable C-module if and only if G has a nonzero endomorphic image of at most countable C-rank.*

Proof. Only the "if"-part needs proving. By 2.2, $G = AF$ with $A \leqslant Q$ and F E-transitive, $E(F) = E(G)$, and C-submodules of F of finite rank are free (3.2). By 2.3, C is a P.I.D. Similarly to [5, p. 119, 87.4] one proves that F is a separable C-module if every pure rank-one submodule of F is a direct summand. Let W be a pure rank-one C-submodule of F. By hypothesis, there exists $0 \neq \varepsilon \in E = E(F) = E(G)$ such that $F\varepsilon$ has at most countable C-rank and, hence, is free. Thus $F \simeq F\varepsilon \oplus \ker \varepsilon$ and $F = U \oplus F'$ for some submodule $U \simeq C$ of F. By 3.4, there exists $\phi \in E$ such that $U = W\phi$, and $\phi \mid W$ is monic by 2.4(ii). Let $\pi \in E$ be the projection onto U along F'. Then $F = W \oplus \ker \phi\pi$. It follows that F is a separable C-module and, thus, so is $G = AF$.

COROLLARY 4.2. *Every E-transitive group G with a non-zero endomorphic image of at most countable C-rank,* $C = \operatorname{Cent} E(G)$, *is strongly homogeneous.*

Proof. 2.3, 4.1, and 3.6

In particular, for modules of countable C-rank, E-transitivity and strong homogeneity coincide: such modules are torsion-free homogeneous completely decomposable C-modules (3.3) and, thus, strongly homogeneous.

Krylov [9] has shown that the center of the endomorphism ring of a countable strongly homogeneous group is an E-ring [13]. In fact, $C = \operatorname{Cent} E(G)$ is an E-ring for every E-transitive group G which is a separable C-module. This follows from

PROPOSITION 4.3. *Let F be a torsion-free E-cyclic E-transitive group and let* $C = \operatorname{Cent} E(F)$. *If F contains a C-submodule of rank one whose additive endomorphisms lift to F then C is an E-ring.*

Proof. Let M be a rank-one C-submodule of F whose additive endomorphisms lift to F. By 3.2, $M = aC$ is cyclic, $a \in F$. Let $\phi\colon aC \to C$ be the map $a\zeta \mapsto \zeta$ for all $\zeta \in C$. Then ϕ is an isomorphism of C-modules. Let $\psi \in E(C^+)$. Then $\phi\psi\phi^{-1} \in E(aC)$ so that $\phi\psi\phi^{-1} = \varepsilon | aC$ for some $\varepsilon \in E$. Hence,

$$\psi = \phi^{-1}(\varepsilon | aC)\,\phi,$$

and ψ is C-linear since both ϕ and $\varepsilon | aC$ are. By [3, p. 199, 1.2(iv)], C is an E-ring.

COROLLARY 4.4. *Let G be an E-transitive group and let* $C = \mathit{Cent}\, E(G)$. *If G is a separable C-module then C is an E-ring.*

This raises

PROBLEM 3. If $C = \operatorname{Cent} E(G)$ for some torsion-free E-transitive group G, must C be an E-ring?

Examples show that not every torsion-free strongly homogeneous integral domain is an E-ring.

REFERENCES

1. D. M. ARNOLD, Strongly homogeneous torsion free abelian groups of finite rank, *Proc. Amer. Math. Soc.* **56** (1976), 67–72.
2. D. M. ARNOLD, "Finite Rank Torsion Free Abelian Groups and Rings," Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1982.

3. R. A. BOWSHELL AND P. SCHULTZ, Unital rings whose additive endomorphisms commute, *Math. Ann.* **288** (1977), 197–214.
4. L. FUCHS, "Infinite Abelian Groups," Vol. I, Academic Press, New York, 1970.
5. L. FUCHS, "Infinite Abelian Groups," Vol. II, Academic Press, New York, 1973.
6. J. HAUSEN, Finite rank torsion-free abelian groups uniserial over their endomorphism rings, *Proc. Amer. Math. Soc.* **83** (1985), 227–231.
7. N. JACOBSON, "Structure of Rings," Amer. Math. Soc., rev. ed., Princeton, N.J., 1968.
8. I. KAPLANSKY, "Infinite Abelian Groups," rev. ed., The University of Michigan Press, Ann Arbor, 1969.
9. P. A. KRYLOV, Strongly homogeneous torsion-free abelian groups, *Sibirsk. Mat. Zh.* **24** (1983), 77–84; English Translation: *Sibirian Math. J.* **24** (1983), 215–221.
10. J. D. REID, On the ring of quasi-endomorphisms of a torsion-free group, "Topics in Abelian Groups," pp. 51–68, Scott, Foresman, Chicago, 1963.
11. J. D. REID, Abelian groups finitely generated over their endomorphism rings, "Abelian Group Theory," Lecture Notes in Mathematics, pp. 41–52, Springer-Verlag, Berlin, 1981.
12. J. D. REID, Abelian groups cyclic over their endomorphism rings, "Abelian Group Theory," Lecture Notes in Mathematics, pp. 191–203, Springer-Verlag, Berlin, 1983.
13. P. SCHULTZ, The endomorphism ring of the additive group of a ring, *J. Austral. Math. Soc.* **15** (1973), 60–69.
14. D. W. SHARPE AND V. VAMOS, "Injective Modules," Cambridge Univ. Press, London, 1972.